Tail Recursion
Review: Evaluating a Function Application

One simple rule: One evaluates a function application $f(e_1, \ldots, e_n)$

- by evaluating the expressions $e_1, \ldots, e_n$ resulting in the values $v_1, \ldots, v_n$, then
- by replacing the application with the body of the function $f$, in which
- the actual parameters $v_1, \ldots, v_n$ replace the formal parameters of $f$. 

This can be formalized as a *rewriting of the program itself*:

\[
\text{def } f(x_1, \ldots, x_n) = B; \ldots f(v_1, \ldots, v_n) \\
\rightarrow \\
\text{def } f(x_1, \ldots, x_n) = B; \ldots [v_1/x_1, \ldots, v_n/x_n] B
\]

Here, \([v_1/x_1, \ldots, v_n/x_n] B\) means:

The expression \(B\) in which all occurrences of \(x_i\) have been replaced by \(v_i\).

\([v_1/x_1, \ldots, v_n/x_n]\) is called a *substitution*. 
Consider \( \text{gcd} \), the function that computes the greatest common divisor of two numbers.

Here’s an implementation of \( \text{gcd} \) using Euclid’s algorithm.

```python
def gcd(a: Int, b: Int): Int =
    if (b == 0) a else gcd(b, a % b)
```
Rewriting example:

\[ \text{gcd}(14, 21) \text{ is evaluated as follows:} \]

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\[ \rightarrow \text{if } (21 == 0) 14 \text{ else gcd}(21, 14 \% 21) \]
Rewriting example:

\[ \text{gcd}(14, 21) \] is evaluated as follows:

\[
\text{gcd}(14, 21) \equiv \begin{cases} 14 & \text{if } (21 == 0) \\ \text{gcd}(21, 14 \mod 21) & \text{else} \end{cases}
\]

\[
\equiv \begin{cases} 14 & \text{if } (\text{false}) \\ \text{gcd}(21, 14 \mod 21) & \text{else} \end{cases}
\]
Rewriting example:

gcd(14, 21) is evaluated as follows:

gcd(14, 21)
→ if (21 == 0) 14 else gcd(21, 14 % 21)
→ if (false) 14 else gcd(21, 14 % 21)
→ gcd(21, 14 % 21)
Rewriting example:

\[ \text{gcd}(14, 21) \text{ is evaluated as follows:} \]

\[ \text{gcd}(14, 21) \]

\[ \rightarrow \text{if } (21 == 0) 14 \text{ else gcd}(21, 14 \mod 21) \]

\[ \rightarrow \text{if } (\text{false}) 14 \text{ else gcd}(21, 14 \mod 21) \]

\[ \rightarrow \text{gcd}(21, 14 \mod 21) \]

\[ \rightarrow \text{gcd}(21, 14) \]
Rewriting example:

\[ \text{gcd}(14, 21) \] is evaluated as follows:

\[ \text{gcd}(14, 21) \]
\[ \rightarrow \text{if } (21 == 0) 14 \text{ else gcd}(21, 14 \mod 21) \]
\[ \rightarrow \text{if } (\text{false}) 14 \text{ else gcd}(21, 14 \mod 21) \]
\[ \rightarrow \text{gcd}(21, 14 \mod 21) \]
\[ \rightarrow \text{gcd}(21, 14) \]
\[ \rightarrow \text{if } (14 == 0) 21 \text{ else gcd}(14, 21 \mod 14) \]
Rewriting example:

gcd(14, 21) is evaluated as follows:

\[ \text{gcd}(14, 21) \]
\[ \rightarrow \text{if } (21 == 0) \text{ 14 else gcd}(21, 14 \% 21) \]
\[ \rightarrow \text{if } (\text{false}) \text{ 14 else gcd}(21, 14 \% 21) \]
\[ \rightarrow \text{gcd}(21, 14 \% 21) \]
\[ \rightarrow \text{gcd}(21, 14) \]
\[ \rightarrow \text{if } (14 == 0) \text{ 21 else gcd}(14, 21 \% 14) \]
\[ \rightarrow \text{gcd}(14, 7) \]
Rewriting example:

gcd(14, 21) is evaluated as follows:

gcd(14, 21)
→ if (21 == 0) 14 else gcd(21, 14 % 21)
→ if (false) 14 else gcd(21, 14 % 21)
→ gcd(21, 14 % 21)
→ gcd(21, 14)
→ gcd(21, 14)
→ if (14 == 0) 21 else gcd(14, 21 % 14)
→ gcd(14, 7)
→ gcd(7, 0)
Rewriting example:

gcd(14, 21) is evaluated as follows:

gcd(14, 21)
→ if (21 == 0) 14 else gcd(21, 14 % 21)
→ if (false) 14 else gcd(21, 14 % 21)
→ gcd(21, 14 % 21)
→ gcd(21, 14)
→ gcd(21, 14)
→ if (14 == 0) 21 else gcd(14, 21 % 14)
→ gcd(14, 7)
→ gcd(7, 0)
→ if (0 == 0) 7 else gcd(0, 7 % 0)
Rewriting example:

gcd(14, 21) is evaluated as follows:

gcd(14, 21)

→ if (21 == 0) 14 else gcd(21, 14 % 21)

→ if (false) 14 else gcd(21, 14 % 21)

→ gcd(21, 14 % 21)

→ gcd(21, 14)

→ gcd(21, 14)

→ if (14 == 0) 21 else gcd(14, 21 % 14)

→ gcd(14, 7)

→ gcd(7, 0)

→ if (0 == 0) 7 else gcd(0, 7 % 0)

→ 7
Another rewriting example:

Consider factorial:

```python
def factorial(n: Int): Int =
    if (n == 0) 1 else n * factorial(n - 1)
```

factorial(4)
Another rewriting example:

Consider factorial:

```python
def factorial(n: Int): Int =
    if (n == 0) 1 else n * factorial(n - 1)
```

factorial(4)

→ if (4 == 0) 1 else 4 * factorial(4 - 1)
Another rewriting example:

Consider factorial:

```python
def factorial(n: Int): Int =
    if (n == 0) 1 else n * factorial(n - 1)
```

factorial(4)

→ if (4 == 0) 1 else 4 * factorial(4 - 1)

→ 4 * factorial(3)
Another rewriting example:

Consider factorial:

```python
def factorial(n: Int): Int = 
    if (n == 0) 1 else n * factorial(n - 1)
```

factorial(4)

→ if (4 == 0) 1 else 4 * factorial(4 - 1)

→ 4 * factorial(3)

→ 4 * (3 * factorial(2))
Another rewriting example:

Consider factorial:

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def factorial(n: Int): Int =
    if (n == 0) 1 else n * factorial(n - 1)
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factorial(4)

→ if (4 == 0) 1 else 4 * factorial(4 - 1)

→ 4 * factorial(3)

→ 4 * (3 * factorial(2))

→ 4 * (3 * (2 * factorial(1)))
Another rewriting example:

Consider factorial:

```python
def factorial(n: int): int =
    if (n == 0) 1 else n * factorial(n - 1)
```

factorial(4)

→ if (4 == 0) 1 else 4 * factorial(4 - 1)

→ 4 * factorial(3)

→ 4 * (3 * factorial(2))

→ 4 * (3 * (2 * factorial(1)))

→ 4 * (3 * (2 * (1 * factorial(0))))
Another rewriting example:

Consider factorial:

```python
def factorial(n: Int): Int =
    if (n == 0) 1 else n * factorial(n - 1)
```

```
factorial(4)
→ if (4 == 0) 1 else 4 * factorial(4 - 1)
→ 4 * factorial(3)
→ 4 * (3 * factorial(2))
→ 4 * (3 * (2 * factorial(1)))
→ 4 * (3 * (2 * (1 * factorial(0))))
→ 4 * (3 * (2 * (1 * 1)))
```
Another rewriting example:

Consider factorial:

```python
def factorial(n: Int): Int =
    if (n == 0) 1 else n * factorial(n - 1)
```

factorial(4)

\[ \rightarrow \text{if } (4 == 0) 1 \text{ else } 4 \times \text{factorial}(4 - 1) \]
\[ \rightarrow 4 \times \text{factorial}(3) \]
\[ \rightarrow 4 \times (3 \times \text{factorial}(2)) \]
\[ \rightarrow 4 \times (3 \times (2 \times \text{factorial}(1))) \]
\[ \rightarrow 4 \times (3 \times (2 \times (1 \times \text{factorial}(0)))) \]
\[ \rightarrow 4 \times (3 \times (2 \times (1 \times 1))) \]
\[ \rightarrow 120 \]

What are the differences between the two sequences?
Tail Recursion

*Implementation Consideration:* If a function calls itself as its last action, the function’s stack frame can be reused. This is called *tail recursion.*

⇒ Tail recursive functions are iterative processes.

In general, if the last action of a function consists of calling a function (which may be the same), one stack frame would be sufficient for both functions. Such calls are called *tail-calls.*
Tail Recursion in Scala

In Scala, only directly recursive calls to the current function are optimized.

One can require that a function is tail-recursive using a @tailrec annotation:

```scala
@tailrec
def gcd(a: Int, b: Int): Int = ...
```

If the annotation is given, and the implementation of gcd were not tail recursive, an error would be issued.
Exercise: Tail recursion

Design a tail recursive version of factorial.