Example: Finding Fixed Points
Finding a fixed point of a function

A number $x$ is called a *fixed point* of a function $f$ if

$$f(x) = x$$

For some functions $f$ we can locate the fixed points by starting with an initial estimate and then by applying $f$ in a repetitive way.

$$x, f(x), f(f(x)), f(f(f(x))), \ldots$$

until the value does not vary anymore (or the change is sufficiently small).
This leads to the following function for finding a fixed point:

```scala
val tolerance = 0.0001
def isCloseEnough(x: Double, y: Double) =
    abs((x - y) / x) / x < tolerance
def fixedPoint(f: Double => Double)(firstGuess: Double) = {
    def iterate(guess: Double): Double = {
        val next = f(guess)
        if (isCloseEnough(guess, next)) next
        else iterate(next)
    }
    iterate(firstGuess)
}
```
Here is a *specification* of the sqrt function:

\[
\text{sqrt}(x) = \text{the number } y \text{ such that } y \times y = x.
\]

Or, by dividing both sides of the equation with \( y \):

\[
\text{sqrt}(x) = \text{the number } y \text{ such that } y = \frac{x}{y}.
\]

Consequently, \( \text{sqrt}(x) \) is a fixed point of the function \( (y \mapsto \frac{x}{y}) \).
First Attempt

This suggests to calculate \( \sqrt{x} \) by iteration towards a fixed point:

```java
def sqrt(x: Double) =
    fixedPoint(y => x / y)(1.0)
```

Unfortunately, this does not converge.

Let's add a `println` instruction to the function `fixedPoint` so we can follow the current value of `guess`:
def fixedPoint(f: Double => Double)(firstGuess: Double) = {
  def iterate(guess: Double): Double = {
    val next = f(guess)
    println(next)
    if (isCloseEnough(guess, next)) next
    else iterate(next)
  }
  iterate(firstGuess)
}

sqrt(2) then produces:

2.0
1.0
2.0
1.0
...
One way to control such oscillations is to prevent the estimation from varying too much. This is done by *averaging* successive values of the original sequence:

```python
def sqrt(x: Double) = fixedPoint(y => (y + x / y) / 2)(1.0)
```

This produces

1.5
1.4166666666666665
1.4142156862745097
1.4142135623746899
1.4142135623746899

In fact, if we expand the fixed point function `fixedPoint` we find a similar square root function to what we developed last week.
Functions as Return Values

The previous examples have shown that the expressive power of a language is greatly increased if we can pass function arguments.

The following example shows that functions that return functions can also be very useful.

Consider again iteration towards a fixed point.

We begin by observing that $\sqrt{x}$ is a fixed point of the function $y \mapsto x / y$.

Then, the iteration converges by averaging successive values.

This technique of stabilizing by averaging is general enough to merit being abstracted into its own function.

```scala
def averageDamp(f: Double => Double)(x: Double) = (x + f(x)) / 2
```
Exercise:

Write a square root function using `fixedPoint` and `averageDamp`. 
def sqrt(x: Double) = fixedPoint(averageDamp(y => x/y))(1.0)

This expresses the elements of the algorithm as clearly as possible.
We saw last week that the functions are essential abstractions because they allow us to introduce general methods to perform computations as explicit and named elements in our programming language.

This week, we’ve seen that these abstractions can be combined with higher-order functions to create new abstractions.

As a programmer, one must look for opportunities to abstract and reuse.

The highest level of abstraction is not always the best, but it is important to know the techniques of abstraction, so as to use them when appropriate.